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Properties of uniform random walks in bounded convex bodies

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Abstract
Properties of constant-speed uniform random walks in bounded convex bodies are presented. Average quantities such as the mean length of the trajectories are expressed only according to the first moments of the chord length distribution. Some analytical results are then extended to the case of purely diffusive random walks. Exact results for convex geometric objects of simple shape in two and three dimensions illustrate our points.

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1. Introduction

Pearson random walks [1], consisting of a sum of n random vectors with the same probability density function, arise in a large variety of physical phenomena depending on the probabilistic law for independent random jumps. For instance, isotropic random flights having fixed length have application in polymer chains in three dimensions [2]; it also concerns planar two-dimensional locomotion problems in biology [3]. An exponential law $1/\lambda \exp[-r/\lambda]$ has application to neutron diffusion processes [4]. An analytical expression for the probability density function of the length of the vector sum has been derived, since the initial work of Rayleigh for the case of constant jumps [5], by Chandrasekhar [6] and Flory [2], in terms of an infinite integral with an oscillatory integrand. However, even though the problem is well understood for Pearson random walks in unbounded spaces and in arbitrary dimensions, very little has been done for such random walks in bounded spaces. Indeed, a process evolving inside a domain with boundaries leads to new difficulties when it escapes from the domain. For such processes, the quantities of importance are the average time spent inside the domain before leaving (first exit time), the mean length of the trajectory (this last quantity requiring the mean length of the last jump) and their corresponding probability density functions. Although escape processes and first exit times are thoroughly covered in the literature in particular for Brownian motion [7], for the vast majority of models based on Markov jump processes no purely analytical solutions for the first exit time exist. In this paper, we report the exact solution for the special case of constant-speed uniform random walks in bounded convex bodies using techniques arising from integral geometry. In particular, we focus on the mean length of the trajectories (or first exit time since the particle travels at constant velocity), the mean number of collisions and related quantities. The paper is organized as follows. After reviewing briefly the concepts and the useful results of integral geometry, we derive within this framework the important statistical quantities of the random walks in term of chord length distributions. Then, exact results illustrate our points in two and three dimensions, followed by a conclusion.

2. Concepts and results from integral geometry: chord length distribution and related quantities

Chord length distributions appear when convex bodies are intercepted by random straight lines [8, 9]. These distribution functions are a powerful tool for the description of the size and shape of the intercepted object. It has applications in various fields such as acoustics [10], ecology [11], image analysis [12], stereology [13] and reactor design [14, 15]. Chord length distributions are also fundamental functions for the characterization of random media [16–18]. Five different sorts of randomness were defined by Coleman [19], and three are relevant in the present paper: mainly the μ-randomness, the ν-randomness and the λ-randomness.

1. *μ-randomness:* for a convex body K in $\mathbb{R}^n$ the μ-chord length distribution is defined as $F_\mu(l) = \text{Prob}\{l(M) \leqslant l : M \cap K \neq \varnothing\}$, measured with the uniform density M of random lines in the sense of the theory of geometric probability [20–22]. $f_\mu(l) = \mathrm{d}F_\mu(l)/\mathrm{d}l$ is the corresponding density function (chord length distribution function (CLD)). This definition is also sometimes called isotropic uniform random chords (IUR chords) since this randomness results if the convex body is exposed to a uniform, isotropic field of straight infinite lines [8].
2. *ν-randomness:* a ν-chord is defined by a point inside K and a direction. Both point and direction are from independent uniform distributions. In many fields of radiation research, the term I-chord randomness or interior radiator randomness is used.
3. *λ-randomness:* a λ-chord is the straight line through two points chosen uniformly and independently in the interior of the convex body.

Chord length distribution functions are different for different types of randomness. In the following we use Kellerer's notation. The index $\rho = (\mu, \nu, \lambda)$ labels the different randomness. Expectation values are labelled in the same way. For instance, $f_\mu(l)$ is the CLD under μ-randomness and $\overline{l_\mu}$ and $\overline{l_\mu^2}$ are the mean and the second moment of the μ-chord distribution function. With this notation, the following results hold [9]:

$$f_\mu(l) = \frac{\overline{l_\mu}}{l} f_\nu(l) \tag{1}$$

and

$$f_\mu(l) = \frac{\overline{l_\mu^{n+1}}}{l^{n+1}} f_\lambda(l). \tag{2}$$

Moreover, the $f_\mu(l)$ distribution satisfies a couple of remarkable relations, the first one being Cauchy's formula,

$$\overline{l_\mu} = (n-1)\sqrt{\pi}\frac{\Gamma[(n-1)/2]}{\Gamma[n/2]}\frac{V(K)}{S(K)} \tag{3}$$

and the second relation concerns the $(n+1)$th moment of the chord,

$$\overline{l_\mu^{n+1}} = \frac{n(n+1)}{\pi^{(n-1)/2}}\Gamma\left[\frac{n+1}{2}\right]\frac{V(K)^2}{S(K)} \tag{4}$$

where $V(K)$ is the volume of K and $S(K)$ its surface, and where Γ denotes the Euler Gamma function (see [23] for a short review of these results and [20] for a complete proof). Chord length distributions are related to other distribution functions of importance in stereology such as the distance distributions between two random points in K or the density distribution of random line segments entirely contained inside K [24]. For the present study, we also need the concept of ray distribution functions largely studied in a series of papers by Enns and Ehlers [25–27] in the fields of applied probabilities and independently introduced by Dixmier in an article on random packing [28]. A ray of length r is defined by the distance of a point inside K to the frontier ∂K of K. Let $G_\rho(r) = \Pr\{|P_1P_2| \leqslant r : P_1 \in K, P_2 \in \partial K\}$ be the distribution function of the rays. Moreover, let $g_\rho(r) = \mathrm{d}G_\rho(r)/\mathrm{d}r$ be the corresponding probability density function (for instance, $\rho = \mu$ corresponds to rays coming from uniform randomness and $\rho = \nu$ corresponds to the ν-randomness, i.e., P_1 is selected uniformly within K and P_2 is the intersection point between a uniform direction from P_1 and ∂K). The last result concerns the characteristic function $\gamma_{0\rho}(r)$ originally introduced by Porod [29] as follows: $\gamma_{0\rho}(r)$ represents the probability that a point at a distance r in an arbitrary direction from a given point inside a convex body K is itself also in K. This function is related to the chord length distribution function by

$$\gamma_{0\rho}(r) = \int_r^\delta \mathrm{d}l\, f_\rho(l)\left(1 - \frac{r}{l}\right) \tag{5}$$

where $\delta = \max(l)$ is the greatest length inside K. See Guinier and Fournet's textbook [30] for a detailed study of the $\gamma_{0\rho}(r)$ function and its properties.

3. Constant-speed uniform random walks

In this section, we consider a special kind of diffusive random walk as follows. First, we select a point uniformly distributed inside K, a convex body in $\mathbb{R}^n$. Then, the particle is allowed to move uniformly within a hypersphere of radius $\Delta \geqslant \delta = \max(\text{chord})$ centred at the point position as shown in figure 1. If the particle is inside K, the next point is selected again according to the same uniform law. The process stops once a position is outside K for the first time. In our model, a particle travels with a constant velocity, which is independent of the jump length. Hence, the time spent inside K is proportional to the total length of the trajectory and, in the following, one will be able to speak equivalently about the time spent inside K or the total length of the trajectory. The total length of the trajectory L is defined as the length of the multiple scattering trajectory from the original point inside K to the first exit through ∂K. Let P_n $(n = 0, 1, \ldots)$ be the probability that a trajectory has n collisions and L_n the length of such a trajectory. $\overline{L_n}$ is the expectation value of this trajectory. The expectation value of the total length of the trajectory, $\overline{L}$, is given by

$$\overline{L} = \sum_{n=0}^{\infty} P_n \overline{L_n} \tag{6}$$

and the mean number of collisions $\overline{N}$ per trajectory is

$$\overline{N} = \sum_{n=0}^{\infty} n P_n. \tag{7}$$

Consequently, $\overline{L}$ and $\overline{N}$ are fully described by the P_n and $\overline{L_n}$. The probabilities $\{P_n\}$ are easy to obtain. Let P be the probability that a particle remains inside K after a collision ($\overline{P} = 1 - P$

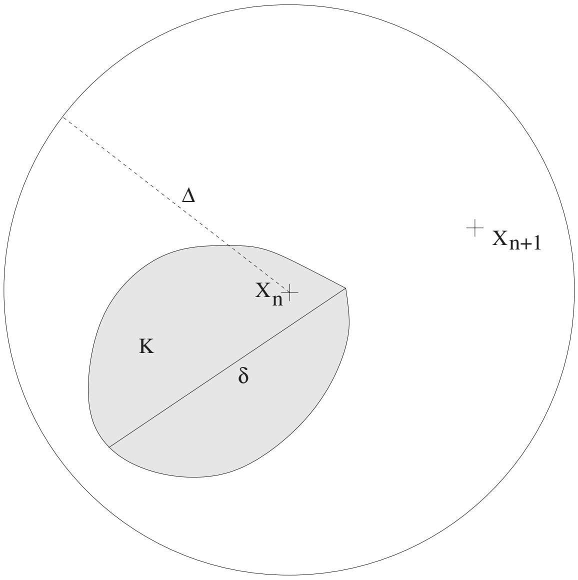

Figure 1. Trajectories are generated from a uniformly distributed point X_n inside a convex body K and the next point X_{n+1} is selected uniformly within a sphere of radius Δ.

is the probability that a particle escapes K after a collision). From the hypothesis of a uniform diffusion inside K, P is the same after each collision; consequently,

$$\begin{cases} P_0 = \overline{P} \\ P_1 = P \times \overline{P} \\ P_n = P^n \times \overline{P}. \end{cases} \tag{8}$$

$\{P_n\}$ are normalized to unity, since $\sum_{n=0}^{\infty} P_n = \overline{P} \times \sum_{n=0}^{\infty} P^n = \overline{P} \times 1/(1-P) = 1$. Moreover,

$$\overline{N} = \sum_{n=0}^{\infty} n P_n = \overline{P} \sum_{n=1}^{\infty} n P^n = \frac{P}{\overline{P}}. \tag{9}$$

Now, from the elementary concept of geometric probability, since the distribution of collisions is uniform within K, P is given by

$$P = \frac{V(K)}{V(B_n(\Delta))} \tag{10}$$

where $V(B_n(\Delta))$ denotes the volume of an n-dimensional sphere of radius Δ, which is

$$V(B_n(\Delta)) = \frac{2\pi^{n/2}}{n\Gamma[n/2]}\Delta^n. \tag{11}$$

Before giving formal results concerning $\overline{L_n}$, P is analysed using elementary results from integral geometry permitting a more intuitive understanding of our future result. First, we need to define the probability density function $p(r)$ of having a jump of length r. Since the new point is selected uniformly within a sphere of radius Δ, and since the direction is uniform,

$p(r)$ is given by

$$p(r) = \frac{S_{n-1}(r)}{V(B_n(\Delta))} \tag{12}$$

where S_{n-1} is the surface area of an $(n-1)$-dimensional sphere of radius r. Furthermore, since $S_{n-1} = 2\pi^{n/2} r^{n-1}/\Gamma[n/2]$, from equation (12) we have

$$p(r) = \frac{n r^{n-1}}{\Delta^n}. \tag{13}$$

Now, the probability density function of having a jump of length r inside K is just the probability density function of having a jump of length r (i.e. $p(r)$) times the probability of being inside K at a distance r from the sphere's centre, which is just the Porod characteristic function $\gamma_{0\nu}(r)$ with ν-randomness (ν-randomness appears since either the origin or the direction of the jump is uniform). By summing up over all r we obtain

$$P = \int_0^{\Delta} \mathrm{d}r\ p(r)\gamma_{0\nu}(r). \tag{14}$$

Substituting equations (5) and (13) into equation (14) leads to

$$P = \int_0^{\Delta} \mathrm{d}l\ f_\nu(l) \int_0^{l} \mathrm{d}r\ \frac{n}{\delta^n} r^{n-1} \left(1 - \frac{r}{l}\right) = \frac{1}{(n+1)\Delta^n} \int_0^{\delta} \mathrm{d}l\ f_\nu(l) l^n \tag{15}$$

and then inserting equation (1) into the preceding equation yields

$$P = \frac{1}{(n+1)\Delta^n \overline{l_\mu}} \int_0^{\delta} \mathrm{d}l\ f_\mu(l) l^{n+1} = \frac{1}{(n+1)\Delta^n} \frac{\overline{l_\mu^{n+1}}}{\overline{l_\mu}}. \tag{16}$$

Finally, substituting equations (3) and (4) into equation (16) we recover the desired result for P, namely equation (10). However, it is worth noting that conversely using equations (10) and (16), and Cauchy's formula (3) allows us to derive equation (4) involving only elementary techniques.

In order to determine the probability distribution of the L_n we first consider the event: the particle makes a jump of length r outside K and we introduce the corresponding probability density function $h(r)$. From the definition of $h(r)$, this conditional probability density function is the product of the two events' probabilities: the particle makes a jump of length r (with density probability $p(r)$) and this jump is outside K (which has probability $1 - \gamma_{0\nu}(r)$), over the probability that a particle makes a jump outside K which is just $\overline{P}$. Thus, $h(r)$ is

$$h(r) = \frac{1}{\overline{P}} \times p(r) \times [1 - \gamma_{0\nu}(r)]. \tag{17}$$

Inserting equations (1) and (5) into the preceding equation gives

$$h(r) = \frac{p(r)}{\overline{P}} \left[1 - \int_r^{\delta} \mathrm{d}l\ f_\nu(l) \left(1 - \frac{r}{l}\right)\right] = \frac{p(r)}{\overline{P}} \left[\frac{r}{\overline{l_\mu}} + \frac{1}{\overline{l_\mu}} \int_0^{r} \mathrm{d}l\ f_\mu(l)(l - r)\right]. \tag{18}$$

Now, let us define the probability density function $h(r, l)$ as the probability density function of having a ray of length l knowing that the particle made a jump of length r outside K (recall that a ray is the distance between a point inside K and a point on ∂K). From the definition of $h(r, l)$, we have $h(r, l) = 0$ for $r > l$ and for $r \leqslant l$ this conditional probability function is also the product of the two events' probabilities:

1. The particle makes a jump of length r outside K which has the density function $h(r)$.
2. Given a jump of length r outside K, the probability of having a ray of length l is $g_\nu(l)/\int_0^r \mathrm{d}u\ g_\nu(u)$, where $g_\nu(l)$ is the ray distribution function with ν-randomness.

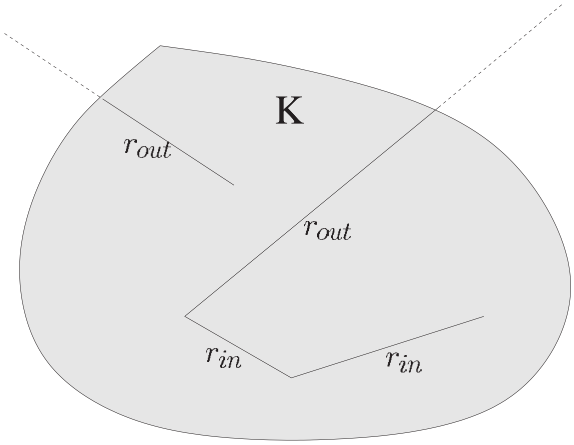

Figure 2. Examples of trajectories with zero and two collisions inside a convex body *K*.

Consequently, $h(r, l)$ is written

$$h(r, l) = h(r) \times \frac{g_\nu(l)}{\int_0^r \mathrm{d}u\, g_\nu(u)} \Theta(r - l) \tag{19}$$

where $\Theta(x)$ is the Heaviside step function. In appendix A it is proved that $g_\mu(l)$ and $g_\nu(l)$ are the same distribution functions; using this result and inserting the expression of $g_\nu(l)$ from equation (A.5) into the integral in the numerator of equation (19) gives

$$\int_0^r \mathrm{d}u\, g_\nu(u) = \int_0^r \mathrm{d}u \frac{1}{\overline{l_\mu}} \int_u^\delta \mathrm{d}l\, f_\mu(l) = \frac{r}{\overline{l_\mu}} + \frac{1}{\overline{l_\mu}} \int_0^r \mathrm{d}l\, f_\mu(l)(l - r). \tag{20}$$

Substituting this last equation and the expression of $h(r)$ (equation (18)) into equation (19) leads to the simplified result

$$h(r, l) = \frac{1}{\overline{P}} p(r) g_\mu(l) \Theta(r - l). \tag{21}$$

From the expresion of $h(r, l)$ it is straightforward to obtain the probability density function $P_{\mathrm{out}}(l)$ of having a last jump of length l,

$$P_{\mathrm{out}}(l) = \int_0^\Delta \mathrm{d}r\, h(r, l) = \frac{1}{\overline{P}} g_\mu(l) \int_l^\Delta \mathrm{d}r\, p(r). \tag{22}$$

Inserting equation (13) into equation (22) gives[1]

$$P_{\mathrm{out}}(l) = \frac{1}{\overline{P}} g_\mu(l) \left(1 - \frac{l^n}{\Delta^n}\right). \tag{23}$$

Consequently, the mean jump outside K, denoted $\overline{r_{\mathrm{out}}}$ (see figure 2), is given by

$$\overline{r_{\mathrm{out}}} = \frac{1}{\overline{P}} \int_0^\delta \mathrm{d}l\, l g_\mu(l) \left(1 - \frac{l^n}{\Delta^n}\right) = \frac{1}{\overline{P}} \left(\overline{r_\mu} - \frac{\overline{r_\mu^{n+1}}}{\Delta^n}\right) \tag{24}$$

[1] A more formal proof of equation (23) is given in appendix B where the κ-randomness corresponding to the randomness of the random walk is introduced.

where $\overline{r_\mu^k} = \int_0^\delta \mathrm{d}l\, l^k g_\mu(l)$ are the kth moments of the μ-ray distribution function. However, Dixmier [28] linked the kth moments of the μ-chords and kth moments of the μ-rays through the general formula $\overline{l_\mu^n} = n\overline{l_\mu}\,\overline{r_\mu^{n-1}}$. Thus, we get

$$\overline{r_{\mathrm{out}}} = \frac{1}{P\overline{l_\mu}}\left(\frac{\overline{l_\mu^2}}{2} - \frac{1}{\Delta^n}\frac{\overline{l_\mu^{n+2}}}{(n+2)}\right). \tag{25}$$

Next, we have to treat the case of a jump inside K. However, due to the Markovian behaviour and the uniformity of the process, the distribution function of the distance between the jumps inside K, $r_{\mathrm{in}}(z)$, is the same as the distribution function of the distance between two random uniformly distributed points in K. Such a distance distribution function has been widely covered in the literature [20–22] and, more precisely, Piefke [31] proved that

$$r_{\mathrm{in}}(z) = B_n z^{n-1}\int_z^\delta \mathrm{d}l\, f_\mu(l)(l-z) \qquad \text{with} \quad n \geqslant 2 \tag{26}$$

where $B_n = S(K)\pi^{(n-1)/2}\left[V(K)^2\Gamma\left[\frac{n+1}{2}\right]\right]^{-1}$ (in particular for the two- and three-dimensional cases: $B_2 = 2S(K)/V(K)^2$ and $B_3 = \pi S(K)/V(K)^2$). An immediate consequence of Piefke's result is that (see again [31] for a complete proof)

$$\overline{r_{\mathrm{in}}} = \frac{B_n}{(n+1)(n+2)}\overline{l_\mu^{n+2}}. \tag{27}$$

These expressions of $\overline{r_{\mathrm{in}}}$ and $\overline{r_{\mathrm{out}}}$ enable us to obtain the mean length $\overline{L_n}$ of trajectories that have exactly n jumps. Indeed, since the process is Markovian, each jump is an independent random variable, thus we have

$$\overline{L_n} = n\overline{r_{\mathrm{in}}} + \overline{r_{\mathrm{out}}}. \tag{28}$$

Inserting equation (28) into equation (6) and doing the summation leads to

$$\overline{L} = \frac{\overline{P}}{P}\overline{r_{\mathrm{in}}} + \overline{r_{\mathrm{out}}}. \tag{29}$$

Finally, inserting the analytical expressions of $\overline{r_{\mathrm{in}}}$ and $\overline{r_{\mathrm{out}}}$ into equation (29) as well as the expression of the coefficients B_n gives

$$\overline{L} = \frac{1}{P}\frac{\overline{l_\mu^2}}{2\overline{l_\mu}} + \left(\frac{\overline{P}}{P}\frac{n}{\overline{l_\mu^{n+1}}} - \frac{1}{P\Delta^n\overline{l_\mu}}\right)\frac{\overline{l_\mu^{n+2}}}{n+2} \tag{30}$$

which is the expression of the mean length of the trajectories expressed only according to the nth moments of the μ-chord length distribution.

4. Exact results for convex objects of simple geometric shapes

In this section, some analytical results are derived for several kinds of convex bodies: a disc and a square for the two-dimensional case and a sphere for the three-dimensional case. For both spherical cases $\Delta = D$ (diameter of the object) and consequently from equation (10), the values of P are $P = 1/4$ for the disc and $P = 1/8$ for the sphere. One has also immediately from equation (9) that the mean number of collisions is $\overline{N} = 1/3$ for the two-dimensional case, and $\overline{N} = 1/7$ for the three-dimensional case. Moreover, the μ-CLD for a sphere of diameter D in arbitrary dimension has been derived by Dixmier [28]. For the two- and three-dimensional cases, the chord distribution functions are given, respectively, by

$$f_\mu(l) = \begin{cases} \dfrac{1}{D^2}\dfrac{l}{\sqrt{1-\frac{l^2}{D^2}}} & (l \leqslant D) \\ 0 & (l > D) \end{cases} \qquad \text{two-dimensional case} \tag{31}$$

Table 1. Simulation results.

	Number of trajectories	$\overline{L}_{\text{analytical}}$	$\overline{L}_{\text{simulation}}$	Variance
Disc	10^9	0.980 866[a]	0.980 885	2.49×10^{-5}
Square	10^9	0.513 422[b]	0.513 414	1.23×10^{-5}

[a] From equation (35).
[b] From equation (37).

and

$$f_\mu(l) = \begin{cases} \dfrac{2l}{D^2} & (l \leqslant D) \\ 0 & (l > D) \end{cases} \qquad \text{three-dimensional case.} \tag{32}$$

From the two preceding expressions of the CLD, the moments of the chord length distribution can be obtained easily, and calculating these moments gives

$$\overline{l_\mu^n} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left[\frac{2+n}{2}\right]}{\Gamma\left[\frac{3+n}{2}\right]} D^n \qquad \text{two-dimensional case} \tag{33}$$

and

$$\overline{l_\mu^n} = \frac{2}{2+n} D^n \qquad \text{three-dimensional case.} \tag{34}$$

Inserting these expressions as well as the value of P into equation (30) gives the mean value of the trajectory,

$$\overline{L} = \begin{cases} \dfrac{208}{135\pi} D & \text{two-dimensional case} \\ \dfrac{99}{245} D & \text{three-dimensional case.} \end{cases} \tag{35}$$

Our last example is devoted to the square which has fewer symmetries. For a square of side a, the CLD is given by [9]

$$f_\mu(l) = \begin{cases} \dfrac{1}{2a} & \text{for} \quad 0 \leqslant l \leqslant a \\ \dfrac{a^2}{l^2\sqrt{l^2 - a^2}} - \dfrac{1}{2a} & \text{for} \quad a < l \leqslant \sqrt{2}a. \end{cases} \tag{36}$$

From equation (36), it is straightforward to calculate the first four moments of the CLD needed for equation (30). One has, respectively, $\overline{l_\mu} = \frac{\pi}{4}a$, $\overline{l_\mu^2} = \left[\frac{1}{3} - \frac{\sqrt{2}}{3} + \log(1+\sqrt{2})\right]a^2$, $\overline{l_\mu^3} = \frac{3}{4}a^3$ and $\overline{l_\mu^4} = \left[\frac{1}{5} + \frac{\sqrt{2}}{10} + \frac{1}{2}\log(1+\sqrt{2})\right]a^4$. Inserting the four previous relations into equation (30) with $\Delta = a\sqrt{2}$ and $P = 1/(2\pi)$ yields

$$\overline{L} = \frac{38 - 41\sqrt{2} + 115\log(1+\sqrt{2})}{15(2\pi - 1)} a. \tag{37}$$

For the disc and the square, Monte Carlo simulations were performed in order to get the behaviour of the distribution function of the total length of the trajectory. Details of the different simulations are presented in table 1 where $\overline{L}$ has already very well converged to the theoretical values given by equations (35) and (37). Figure 3 presents the distribution function of the trajectories; each curve has a tail compared to the pure ray distribution function that comes from the possibility of multiple scattering.

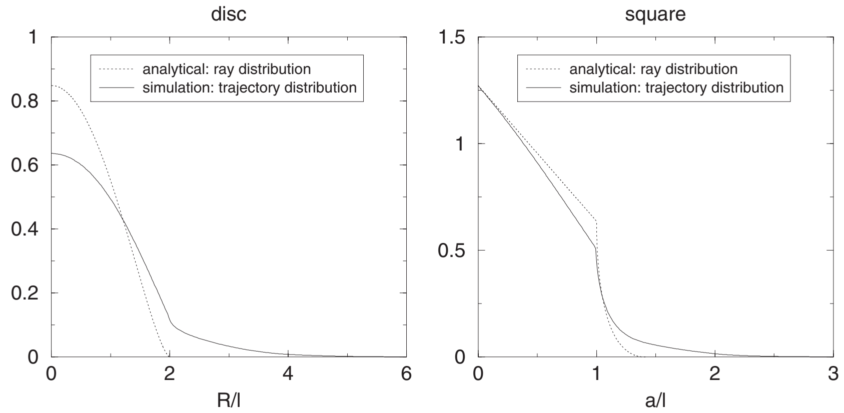

Figure 3. Trajectory and ray distribution functions versus dimensionless distance for a unit disc and for a unit square.

Other analytical results such as the density function $P_{\text{out}}(l)$ given by equation (23) are accessible. An example is given for a square of side a. From equations (A.1) and (36), the ray distribution function can be easily calculated,

$$g_\mu(l) = \frac{4}{\pi a} \times \begin{cases} 1 - \dfrac{l}{2a} & \text{for} \quad 0 \leqslant l \leqslant a \\ \dfrac{l}{2a} - \sqrt{1 - \dfrac{a^2}{l^2}} & \text{for} \quad a < l \leqslant \sqrt{2}a. \end{cases} \tag{38}$$

Then, inserting the preceding equation into equation (23) leads to

$$P_{\text{out}}(l) = \frac{4}{a(\pi - a^2/\Delta^2)} \left(1 - \frac{l^2}{\Delta^2}\right) \times \begin{cases} 1 - \dfrac{l}{2a} & \text{for} \quad 0 \leqslant l \leqslant a \\ \dfrac{l}{2a} - \sqrt{1 - \dfrac{a^2}{l^2}} & \text{for} \quad a < l \leqslant \sqrt{2}a. \end{cases} \tag{39}$$

The analytical behaviour of $P_{\text{out}}(l)$ is presented in figure 4 for two values of Δ.

5. Extension

In this section, we extend our results to the case of trajectories generated from an entry point to the first exit point. More precisely, the entry point is selected uniformly on the boundary of the convex body K. Then, the first jump occurs uniformly in K (the probability that the particle has its first jump inside K is 1). Therefore, results regarding the P_n (equation (8)) as well as those of the mean number of collisions (equation (9)) remain unchanged. However, in order to get the mean value of the total length of the trajectory, we need to calculate the mean value of this first jump inside K (after this first jump, the process is in a state that corresponds to the previous study since the particle is uniformly distributed within K). In other words, we have to calculate the mean distance between two points, one on the boundary of K and the other within K. However, this kind of randomness involving the selection of a random point on ∂K (called the surface radiator randomness or S-randomness) has attracted less interest since it has no known relation to the other kinds of randomness [34]. Nevertheless, the problem has

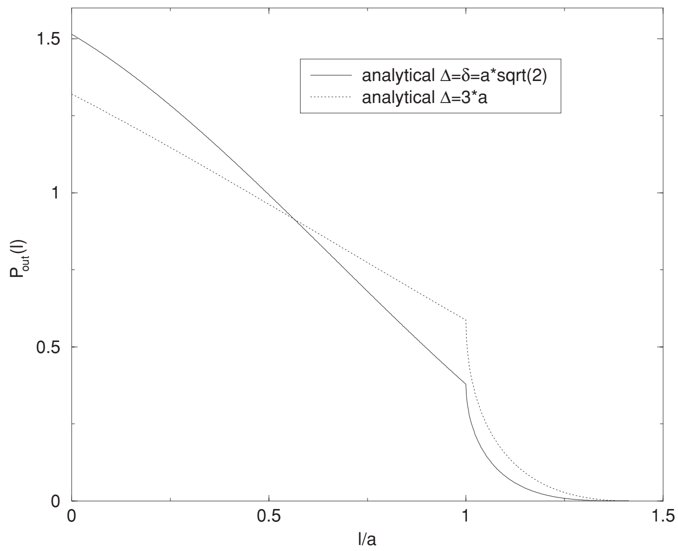

Figure 4. $P_{\text{out}}(l)$ distribution functions versus dimensionless distance l/a for a square of side a.

been solved for some elementary shapes, and Matai [32] gives the following result for the circle:

$$\overline{d} = \frac{16}{9\pi} D. \tag{40}$$

Adding this last distance to the mean length of the trajectories of section 3 gives the mean length of the trajectories for the full process.

Of more interest is the following process when we consider a particle entering the convex body with an isotropic uniform incidence. This hypothesis corresponds to an isotropic incident flux in reactor physics [23]. Moreover, it also corresponds to the hypothesis leading to the invariance property of diffusive random walks recently discovered by Blanco and Fournier [35]. In the following, we consider a particle entering a convex body in $\mathbb{R}^n$ and making a jump of length r according to the probabily density $p(r)$ (in the case studied by Blanco and Fournier $p(r)$ has an exponential law $p(r) = 1/\lambda \exp[-r/\lambda]$, where λ is the mean free path). In the continuation of this section, we calculate the distribution function of trajectories that have no collision (i.e. first jump is out of K) as well as the distribution function of the first jump inside K and their corresponding mean values. For this, we introduce the function $\gamma_1(r)$ defined as follows. $\gamma_1(r)$ represents the probability that a point at a distance r from a given point on ∂K in an arbitrary direction directed inside K is itself in K (see figure 5). Note that we do not consider points at a distance r in the direction directed towards the outside of the body since in this case the probability of having the points in K is trivially 0. This definition is similar to that of the Porod function defined in section 2 when the point considered belongs to the surface of the convex body. From this definition $\gamma_1(r) = 1$ if the point belongs to a chord of length greater than r, and $\gamma_1(r) = 0$ if not. Consequently,

$$\gamma_1(r) = \int_r^{\delta} \mathrm{d}l\, f_\mu(l) \tag{41}$$

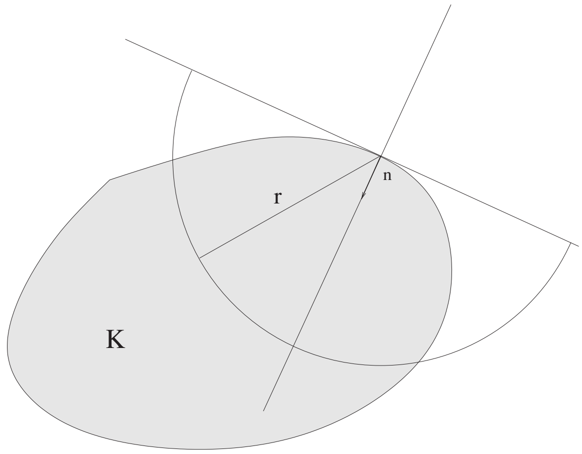

Figure 5. Only points directed towards the interior of K contribute to the definition of $\gamma_1(r)$.

which is, up to a constant, the ray distribution function defined by Dixmier in [28]. With the definition of $\gamma_1(r)$, let us do again the reasoning of section 2 with $\gamma_1(r)$ instead of $\gamma_0(r)$ except that the measure considered is now uniform (μ-randomness). Therefore, we have

$$P = \int_0^{\Delta} \mathrm{d}r\; p(r)\gamma_1(r) = \int_0^{\delta} \mathrm{d}u\; f_\mu(u) \int_0^{u} \mathrm{d}r\; p(r) \tag{42}$$

and

$$h(r) = \frac{1}{P} \times p(r) \times [1 - \gamma_1(r)] = \frac{p(r)}{P} \int_0^{r} \mathrm{d}u\; f_\mu(u). \tag{43}$$

The conditional distribution function $h(r, l)$ is again given by equation (19) except that the chords play the role of the rays; consquently $h(r, l)$ may be written as

$$h(r, l) = h(r) \times \frac{f_\mu(l)}{\int_0^r \mathrm{d}u\; f_\mu(u)} \Theta(r - l) = \frac{1}{P} p(r) f_\mu(l) \Theta(r - l) \tag{44}$$

and the distribution functions of jumps that are outside and inside K at the first step are given respectively by

$$P_{\text{out}}(l) = \int_0^{\infty} \mathrm{d}r\; h(r, l) = \frac{1}{P} f_\mu(l) \int_l^{\infty} \mathrm{d}r\; p(r) \tag{45}$$

and

$$P_{\text{in}}(l) = \frac{p(l)}{P} \gamma_1(l). \tag{46}$$

In order to illustrate our point, the convex body is taken as a sphere and we consider two special types of random walks. The first is a constant jump random walk of length λ whose distribution function is $p(r) = \delta(r - \lambda)$ and the second is the case studied by Blanco and Fournier corresponding to $p(r) = 1/\lambda \exp[-r/\lambda]$. Recalling that the μ-CLD for the sphere is given by equation (32), after a little algebra, we obtain for the constant jump random walk

$$\begin{cases} P = 1 - \dfrac{\lambda^2}{D^2} \\ P_{\text{out}}(l) = \dfrac{2l}{\lambda^2} \Theta(\lambda - l) \\ P_{\text{in}}(l) = \delta(l - \lambda) \end{cases} \tag{47}$$

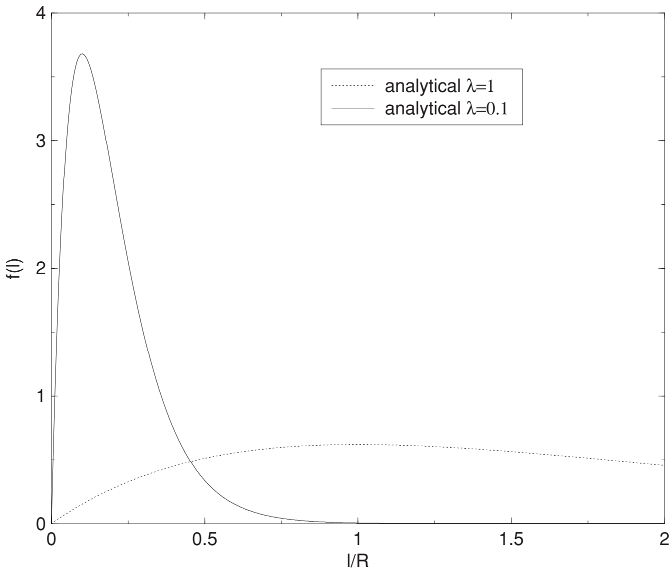

Figure 6. $P_{\mathrm{out}}(l)$ distribution functions versus dimensionless distance l/R for a sphere of radius R.

where λ is understood to be smaller than the sphere's diameter. For the exponential case we obtain

$$\begin{cases} P = 1 - \frac{2\lambda^2}{D^2} + \frac{2\lambda}{D^2}(D+\lambda)\,\mathrm{e}^{-D/\lambda} \\ P_{\mathrm{out}}(l) = \frac{l\,\mathrm{e}^{-l/\lambda}}{\lambda^2(1-(1+D/\lambda)\,\mathrm{e}^{-D/\lambda})} \\ P_{\mathrm{in}}(l) = \frac{\mathrm{e}^{-l/\lambda}}{\lambda}\frac{1-\frac{l^2}{D^2}}{1-\frac{2\lambda^2}{D^2}+\frac{2\lambda}{D^2}(D+\lambda)\,\mathrm{e}^{-D/\lambda}}. \end{cases} \tag{48}$$

Analytical results for the exponential case are presented in figure 6 for two different values of λ.

6. Conclusion

Techniques from geometrical probabilities or integral geometry make it possible to study some special types of random walks in convex bodies. In our studies, we restricted ourselves to uniform random walks where the hypothesis of having at each step a position uniformly distributed within the body is essential to apply the results of integral geometry directly. Nevertheless, some preliminary results have been obtained for diffusive random walks in convex bodies. Unfortunately, even for the simple case of an exponential probability density law for the jump process, studying the whole diffusive random walk system remains an open problem in the general context of diffusive random walks in bounded spaces [33]. Indeed, due to this exponential form of the jump process, the points of diffusion are not uniformly distributed within the body, and consequently the reasoning of section 3 cannot be directly applied. However, we think that combining our approach, which is technically simple, with purely geometric processes such as Poissonian random lines processes is a promising approach to study rigorously diffusive random walks in bounded spaces.

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Appendix A. Equivalence of the μ- and ν-ray distribution functions

In this appendix, we prove the equivalence between the ν-ray distribution function and the ray distribution function introduced by Dixmier [28] which corresponds, as we will see shortly, to the μ-ray distribution function. This will enable us to use equivalently results from geometric probability and Dixmier results (named radius distribution function in [28]). Indeed, Dixmier considers rays supported by a random chord in the sense of geometric probability, i.e. in the vocabulary of this paper μ-chord and consequently Dixmier studied μ-ray distribution functions. Moreover, he established that since the rays of length r are supported by the chords of length greater than r, the distribution of rays $g_\mu(r)$ is related to the distribution of random chords $f_\mu(l)$ by

$$g_\mu(r) = \frac{1}{\overline{l_\mu}} \int_r^\delta \mathrm{d}u \, f_\mu(u) \tag{A.1}$$

where $\delta = \max(l)$ as usual.

In the following, let K be a convex body in $\mathbb{R}^n$, $V(K)$ indicate its volume and $S(K)$ its surface. On one hand, Enns and Ehlers [26] derived the relation

$$1 - G_\nu(l) = \frac{V(K)}{n\omega_n l^{n-1}} d(l) \tag{A.2}$$

where $d(l)$ is the distance between two points chosen uniformly and independently in the interior of K. On the other hand, as we already mentioned in section 3, Piefke [31] showed that

$$d(l) = B_n l^{n-1} \int_l^\delta \mathrm{d}u \, f_\mu(u)(u - l) \qquad \text{with} \quad n \geqslant 2 \tag{A.3}$$

where $B_n = S(K)\pi^{(n-1)/2}\left[V(K)^2\Gamma\left[\frac{n+1}{2}\right]\right]^{-1}$. By putting the two preceding relations together, one obtains

$$1 - G_\nu(l) = \frac{1}{\overline{l_\mu}} \int_l^\delta \mathrm{d}u \, f_\mu(u)(u - l) \tag{A.4}$$

where we have used Cauchy's formula (equation (3)) by introducing $\overline{l_\mu}$. Differentiating the preceding equation yields

$$g_\nu(l) = \frac{1}{\overline{l_\mu}} \int_l^\delta \mathrm{d}u \, f_\mu(u) \tag{A.5}$$

which is precisely the $g_\mu(r)$ ray distribution function of Dixmier (equation (A.1)).

Appendix B. Chord length distribution and ray distribution

In this appendix, we derive the chord length distribution and the ray distribution generated by the random walk process described in section 3 when the first point P is uniformly distributed inside a convex body K and the second point Q is uniformly distributed outside K in a sphere of radius Δ centred on the first point as shown in figure 7. More precisely, we want to express the CLD generated by the random lines PQ in term of CLD corresponding to the μ-randomness.

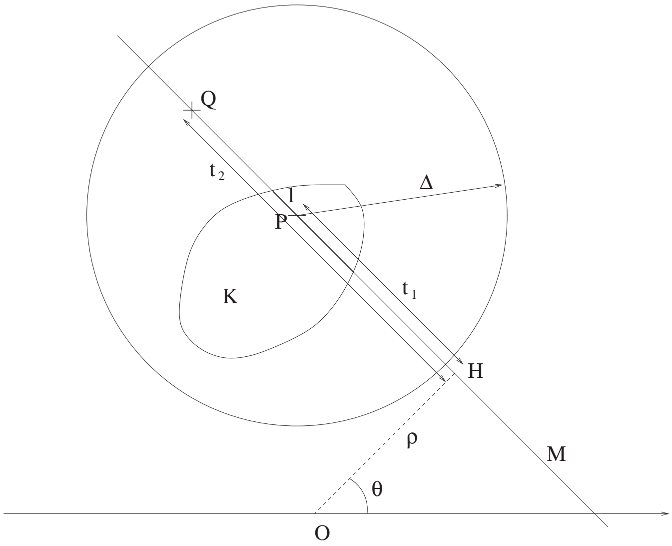

Figure 7. Two random points P and Q uniformly distributed within K and $B_\Delta \backslash K$ define the κ-randomness.

For simplicity, we initially treat the two-dimensional case. Consider the perpendicular OH from the origin O to the line passing through P and Q. Let θ be the angle this perpendicular makes with the x-axis and ρ the distance of the line to the origin (see figure 7). Then, the pair of points P and Q can also be determined by ρ, θ, t_1 and t_2, where t_1 and t_2 are the distances of P and Q from H, respectively. With this notation, from Matai [32] (p 92) we have the following lemma. Let M be the line PQ and $\mathrm{d}M = \mathrm{d}\rho\,\mathrm{d}\theta$ the element of the invariant measure for M (μ-randomness), then

$$\mathrm{d}P\,\mathrm{d}Q = |t_2 - t_1|\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}M. \tag{B.1}$$

Integrating over t_1 and t_2 leads to

$$\begin{aligned}\int |t_2 - t_1|\,\mathrm{d}t_1\,\mathrm{d}t_2 &= \int_0^l \mathrm{d}t_1 \left[\int_l^{t_1+\Delta} \mathrm{d}t_2(t_2 - t_1) + \int_{-\Delta+t_1}^{l} \mathrm{d}t_2(t_1 - t_2)\right] \\ &= \int_0^l \mathrm{d}t_1\left[-t_1^2 + 2t_1 l + \Delta^2 - l^2\right] \\ &= \Delta^2 l - \frac{l^3}{3}.\end{aligned} \tag{B.2}$$

The right-hand side of equation (B.1) reduces to $(\Delta^2 l - l^3/3)\,\mathrm{d}M$ where $\mathrm{d}M$ is the measure of the μ-randomness. Hence, the relationship between the μ-random density $f_\mu(l)$ and the density of our case denoted by $f_\kappa(l)$ is the following:

$$f_\kappa(l) = a\left(\Delta^2 l - \frac{l^3}{3}\right) f_\mu(l) \tag{B.3}$$

where a is a normalization constant. Integrating over l and using relations (3) and (4) with $n = 2$ gives

$$a = \frac{1}{\int_0^\infty \mathrm{d}l\,\left(\Delta^2 l - \frac{l^3}{3}\right) f_\mu(l)} = \frac{1}{\Delta^2 \overline{l_\mu} - \frac{1}{3}\overline{l_\mu{}^3}} = \frac{L(K)}{S(K)(B_\Delta - S(K))} \tag{B.4}$$

where $S(K)$ is the surface of K and $L(K)$ is its perimeter and where $B_\Delta = \pi\Delta^2$ is the surface of a disc of radius Δ. Thus, $f_\kappa(l)$ is finally given by

$$f_\kappa(l) = \frac{L(K)}{S(K)(B_\Delta - S(K))}\left(\Delta^2 l - \frac{l^3}{3}\right) f_\mu(l). \tag{B.5}$$

The normalization constant a has a simple probabilistic meaning: factor $1/[S(K)(B_\Delta - S(K))]$ is due to the joint density of P and Q since P is uniform within K and Q is uniform within $B_\Delta\backslash K$. The last factor $L(K)$ comes from the normalization of the μ-random chords which is precisely equal to the perimeter of the convex body [20]. Inserting equations (1) and (2) with $n = 2$ into the preceding equation gives the expression of $f_\kappa(l)$ in a more symmetric form,

$$f_\kappa(l) = \frac{B_\Delta f_\nu(l) - S(K) f_\lambda(l)}{B_\Delta - S(K)}. \tag{B.6}$$

In the limit of large Δ, we get $\lim_{\Delta\to\infty} f(l) = f_\nu(l)$ which is the desired result. Indeed, in this case $S(K)$ becomes negligible compared to B_Δ; the direction of PQ is then uniform corresponding to the ν-randomness hypothesis.

Generalization to the n-dimensional case is straightforward using Santalo's results, since in $\mathbb{R}^n$ the density of points may be written as (see Santalo [20], p 237)

$$\mathrm{d}P\,\mathrm{d}Q = |t_2 - t_1|^n\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}M \tag{B.7}$$

where again t_1 and t_2 are the abscissas of P and Q on M. Integrating over t_1 and t_2 leads to

$$\int |t_2 - t_1|^n\,\mathrm{d}t_1\,\mathrm{d}t_2 = \frac{2}{n}\left(\Delta^n l - \frac{l^{n+1}}{n+1}\right) \tag{B.8}$$

and the normalized κ-chord length distribution is

$$f_\kappa(l) = \frac{(n+1)\Delta^n l - l^{n+1}}{(n+1)\Delta^n \overline{l_\mu} - \overline{l_\mu{}^{n+1}}} f_\mu(l) \tag{B.9}$$

or, by introducing the ν- and λ-chord densities as before,

$$f_\kappa(l) = \frac{B_\Delta f_\nu(l) - V(K) f_\lambda(l)}{B_\Delta - V(K)} \tag{B.10}$$

where $B_\Delta = \omega_n\Delta^n$ is the volume of the n-dimensional sphere of radius Δ ($\omega_n = 2\pi^{n/2}/\Gamma[n/2]n$) and where $V(K)$ is the volume of K. From equation (B.10), we can say that the randomness of the last jump or the κ-randomness has two parts. The first comes from a ν-randomness (with coefficient $B_\Delta/(B_\Delta - V(K))$) and the second comes from a λ-randomness (with coefficient $-V(K)/(B_\Delta - V(K))$). Consequently, the κ-ray distribution also has two parts, coming from the ν- and λ-randomness, respectively. More precisely,

$$g_\kappa(l) = \frac{B_\Delta g_\nu(l) - V(K) g_\lambda(l)}{B_\Delta - V(K)}. \tag{B.11}$$

However, in [26] it is established that

$$g_\lambda(l) = \frac{\omega_n l^n}{V(K)} g_\nu(l) \tag{B.12}$$

thus, equation (B.11) may be written as

$$g_\kappa(l) = \frac{B_\Delta - \omega_n l^n}{B_\Delta - V(K)} g_\nu(l). \tag{B.13}$$

Recalling that $B_\Delta = \omega_n\Delta^n$ and that $g_\nu(l) = g_\mu(l)$, one obtains

$$g_\kappa(l) = \frac{B_\Delta}{B_\Delta - V(K)}\left(1 - \frac{l^n}{\Delta^n}\right) g_\mu(l) \tag{B.14}$$

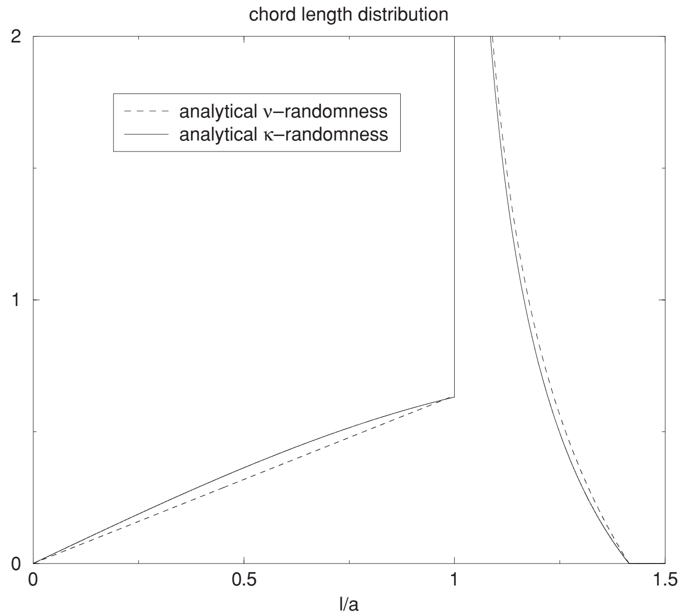

Figure 8. Chord length distribution functions versus dimensionless distance l/a for a square of side a.

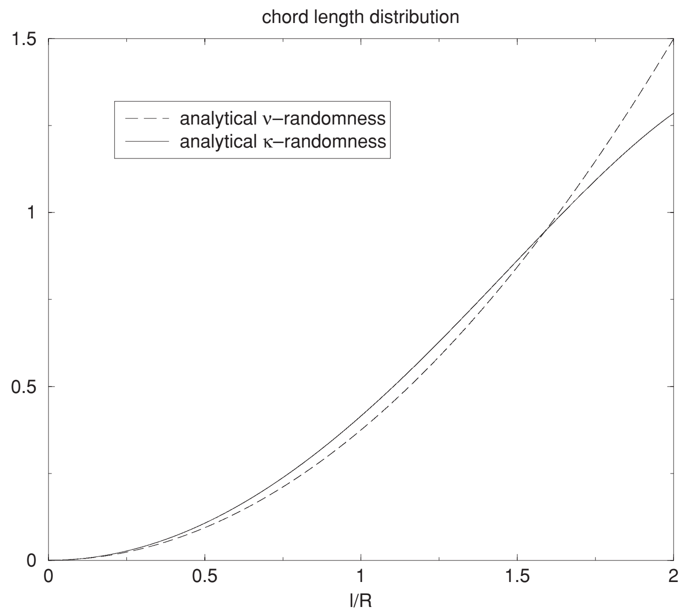

Figure 9. Chord length distribution functions versus dimensionless distance l/R for a sphere of radius R.

and since the coefficient $B_\Delta/(B_\Delta - V(K))$ is just $1/\overline{P}$, we recover the distribution function of the last jump, namely equation (23).

Examples of κ-chord length distribution are presented for the square in figure 8 and for the sphere in figure 9.

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